

# Further Results of the Cryptographic Properties on the Butterfly Structures

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## Abstract

Recently, a new structure called butterfly introduced by Perrin et al. is attractive for that it has very good cryptographic properties: the differential uniformity is at most equal to 4 and algebraic degree is also very high when exponent  $e = 3$ . It is conjectured that the nonlinearity is also optimal for every odd  $k$ , which was proposed as an open problem.

In this paper, we further study the butterfly structures and show that these structures with exponent  $e = 2^i + 1$  have also very good cryptographic properties. More importantly, we prove in theory the nonlinearity is optimal for every odd  $k$ , which completely solve the open problem. Finally, we study the butterfly structures with trivial coefficient and show these butterflies have also optimal nonlinearity. Furthermore, we show that the closed butterflies with trivial coefficient are bijective as well, which also can be used to serve as a cryptographic primitive.

*Keywords:*

S-box, APN, butterfly structure, permutation, differential uniformity, nonlinearity

*MSC:* 94A60, 11T71, 14G50

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## 1. Introduction

Many block ciphers use substitution boxes (S-boxes) to serve as the confusion part. To obtain a correct decryption, S-boxes are usually chosen to be permutation over a finite field with characteristic 2 and even extension degree, i.e.,  $\mathbb{F}_{2^{2k}}$ . For ease of implementation and to have good cryptographic properties to resist various kinds of cryptographic attacks, S-boxes used in block ciphers should possess low differential uniformity to resist differential attack [3], high nonlinearities to resist linear attack [10].

It is well known that for any function defined over  $\mathbb{F}_{2^n}$ , the lowest differential uniformity is 2, and the functions achieving this value are called almost perfect nonlinear (APN) functions. Unfortunately, it is very hard to construct APN permutations for  $n$  even. Up to now, only one APN permutation over  $\mathbb{F}_{2^6}$  has been found. To find any other APN permutations over  $\mathbb{F}_{2^n}$  for even  $n$  is called the BIG APN problem [7].

Therefore, a natural tradeoff method is to use differentially 4-uniform permutations as S-boxes. For instance, the AES (advanced encryption standard) uses a differentially 4-uniform

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function, namely, the inverse function as S-boxes. Hence to provide more choices for the S-boxes, it is of significant importance to construction more infinite classes differentially 4-uniform permutations over  $\mathbb{F}_{2^{2m}}$  with good cryptographic properties.

Recently in [9], Perrin et al. introduced a new structure called butterfly structure and showed that these structures with exponent  $e = 3 \times 2^t$  always have differential uniformity at most 4 when  $n = 2k$  with  $k$  odd. The authors also verified experimentally that the nonlinearity of the butterfly structure is equal to  $2^{2k-1} - 2^k$  for  $k = 3, 5, 7$ . However, they could not prove it in the general case and conjecture that equality is true for every odd  $k$ .

In [9], Li and Wang proposed a construction with 3-round Feistel structure, which is actually a particular cases of the butterfly structure with coefficient 1. They proved that this structure have differential uniformity 4 and algebraic degree  $k$ .

In general, the cost of hardware implementation of nonlinear functions is increasing with its input and output size. Thus implementing functions over subfield often cost much less than implementing functions over the larger field. It is an huge advantage of constructing S-boxes over  $\mathbb{F}_{2^k}$  with butterfly structure for that we only need to implement the exponent functions over  $\mathbb{F}_{2^k}$ . Therefore, comparing with  $2k$ -bit S-boxes constructed directly with permutation over  $\mathbb{F}_{2^{2k}}$ , S-boxes over  $\mathbb{F}_{2^k}$  constructed via butterfly structure with exponent function over  $\mathbb{F}_{2^k}$  cost much less in hardware implementation.

In the present paper, we further revisit the butterfly structure and prove that these structures with exponent  $e = (2^i + 1) \times 2^t$  also have differential uniformity at most 4 when  $n = 2k$  with  $k$  odd. Moreover, we prove theoretically that the nonlinearity equality is true for every odd  $k$ , which means these constructions have the optimal nonlinearity in the sense that no known functions of a field of even size have a higher nonlinearity. Finally, we also study the butterfly structure with trivial coefficient  $\alpha = 1$ , and show that nonlinearity are also optimal. Besides, the closed butterfly structure with trivial coefficient are also a permutation, which also can be used to serve as a cryptographic primitive.

The rest of this paper is organized as follows. In the next section, we recall needed knowledge and some necessary definitions and results. In Section 3, we show that the differential uniformity of butterfly structures with branch size  $k$  odd, exponent  $e = (2^i + 1) \times 2^t$  and nontrivial coefficient are at most equal to 4, and the nonlinearity are optimal as well. In section 4, we further study the butterfly structure with coefficient  $\alpha = 1$  and show that the structures have also the optimal nonlinearity. The proof of bijective of closed butterfly is also given in this section. The concluding remarks are given in Section 5.

## 2. Preliminaries

Throughout this paper, let  $n$  is a positive integer,  $\mathbb{F}_{2^n}$  be the finite field with  $2^n$  elements and  $\mathbb{F}_{2^n}^*$  be the multiplicative group of order  $2^n - 1$ . A function  $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$  can be represented uniquely in a polynomial form in  $\mathbb{F}_{2^n}[x]/\langle x^{2^n} + x \rangle$  as

$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

For any  $l$ ,  $0 \leq l \leq 2^n - 1$ , the number  $w_2(l)$  of the nonzero coefficients  $l_j \in \mathbb{F}_2$  in the binary expansion  $l = \sum_{j=0}^{n-1} l_j 2^j$  is called the 2-weight of  $l$ . The algebraic degree of  $F$ , denoted by  $\deg(F)$ , is equal to the maximum 2-weight of  $i$  such that  $c_i \neq 0$ .

**Definition 1 ([6]).** For a function  $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ , the differential uniformity of  $F(x)$  is denoted as

$$\delta_F = \max\{\delta_F(a, b) : a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}\},$$

where  $\delta_F(a, b) = |\{x \in \mathbb{F}_{2^n} : F(x+a) + F(x) = b\}|$ . The differential spectrum of  $F(x)$  is the set

$$\{\delta_F(a, b) : a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}\}.$$

$F(x)$  is called differential  $\delta$ -uniform if  $\delta_F = \delta$ . It is easy to see that if  $x_0$  is a solution of  $F(x+a) + F(x) = b$ , so does  $x_0 + a$ . Thus the lower bound on differential uniformity of  $F(x)$  is 2. The functions which achieve this bound are called almost perfect nonlinear functions (APN).

For any function  $F : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$ , we define the Walsh transform of  $F$  as

$$\mathcal{W}_F(a, b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\text{Tr}(bF(x) + ax)}, \quad a, b \in \mathbb{F}_{2^n},$$

where  $\text{Tr}(x) = x + x^2 + \dots + x^{2^{n-1}}$  is the absolute trace function from  $\mathbb{F}_{2^n}$  to  $\mathbb{F}_2$ . The set  $\Lambda_F = \{\mathcal{W}_F(a, b) : a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^*\}$  is called the Walsh spectrum of the function  $F$ .

The nonlinearity of  $F$  is defined as

$$\mathcal{NL}(F) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^*} |\mathcal{W}_F(a, b)|.$$

It is known that if  $n$  is odd the nonlinearity of  $F$  satisfies the inequality  $\mathcal{NL}(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}$  [4] and in case of equality  $F$  is called almost bent (AB). The notion of AB function is closely connected the notion of APN function. AB function exist only for  $n$  odd and oppose an optimum resistance to linear cryptanalysis. Besides, every AB function is APN, and in the  $n$  odd case, any quadratic APN function is AB function. A comprehensive survey on APN and AB functions can be found in [1, 2].

While  $n$  is even, the upper bound of nonlinearity is still open. The known maximum nonlinearity is  $2^{n-1} - 2^{\frac{n}{2}}$  [5]. It is conjectured that  $\mathcal{NL}(F)$  is upper bounded by  $2^{n-1} - 2^{\frac{n}{2}}$ . These functions which meet this bound are usually called optimal (maximal) nonlinear functions.

For two functions  $F, G : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n}$  are called extended affine equivalent (EA-equivalent), if  $G(x) = A_1(F(A_2(x))) + A_3(x)$ , where  $A_1(x)$  and  $A_2(x)$  are affine permutations on  $\mathbb{F}_{2^n}$  and  $A_3(x)$  is an affine function over  $\mathbb{F}_{2^n}$ . They are called CCZ-equivalent (Carlet-Charpin-Zinoviev equivalent) if there is an affine permutation over  $\mathbb{F}_{2^n}^2$  which maps  $\mathcal{G}_F$  to  $\mathcal{G}_G$ , where  $\mathcal{G}_F = \{(x, F(x)) : x \in \mathbb{F}_{2^n}\}$  is the graph of  $F$ , and  $\mathcal{G}_G$  is the graph of  $G$ .

It is well known that EA-equivalence implies CCZ-equivalence, but not vice versa. Differential uniformity, nonlinearity and Walsh spectrum are invariants of both EA-equivalence and CCZ-equivalence. Algebraic degree is preserved by EA-equivalence, but not CCZ-equivalence. However, neither EA-equivalence nor CCZ-equivalence preserve permutations.

**Definition 2 ([9]).** Let  $k$  be an integer and  $\alpha \in \mathbb{F}_{2^k}$ ,  $e$  be an integer such that  $x \mapsto x^e$  is a permutation over  $\mathbb{F}_{2^k}$  and  $R_z[e, \alpha](x) = (x + \alpha z)^e + z^e$  be the keyed permutation. The Butterfly Structures are defined as follows:

(a) the Open Butterfly Structure with branch size  $k$ , exponent  $e$  and coefficient  $\alpha$  is the function denoted  $H_e^\alpha$  defined by:

$$H_e^\alpha(x, y) = (R_{R_y^{-1}[e, \alpha](x)}(y), R_y^{-1}[e, \alpha](x)),$$

(b) the Closed Butterfly Structure with branch size  $k$ , exponent  $e$  and coefficient  $\alpha$  is the function denoted  $V_e^\alpha$  defined by:

$$V_e^\alpha(x, y) = (R_x[e, \alpha](y), R_y[e, \alpha](x)).$$

Note that  $H_e^\alpha$  always a permutation over  $\mathbb{F}_{2^k}^2$ , while  $V_e^\alpha$  maybe not. In fact,  $H_e^\alpha$  is an involution over  $\mathbb{F}_{2^k}^2$ , which means  $H_e^\alpha(H_e^\alpha(x, y)) = (x, y)$ . Furthermore, the permutation  $H_e^\alpha$  and the function  $V_e^\alpha$  are CCZ-equivalent [9].

The Walsh spectrum and algebraic degree of a function  $F(x) \in \mathbb{F}_{2^n}[x]$  have the following containment relationships, which is needed to proof our results.

**Lemma 1 ([2]).** *Suppose  $F(x) \in \mathbb{F}_{2^n}[x]$ . If  $2^l | \mathcal{W}_F(a, b)$  for any  $a, b \in \mathbb{F}_{2^k}$  with  $b \neq 0$ , then the algebraic degree of  $F(x)$  is at most equal to  $n - l + 1$ .*

Let  $L$  be a extension of field  $K$ ,  $\sigma \in \text{Gal}(L/K)$ , and polynomial  $w(t) = \sum_{j=0}^l c_j t^j \in L[t]$ . Then  $w(t)$  acts on a element  $x$  of  $L$  is defined as  $w(\sigma)x = \sum_{j=0}^l c_j \sigma^j(x)$ . The following lemma is also needed.

**Lemma 2 ([11]).** *Let  $L$  be a cyclic Galois extension of  $K$  of degree  $n$  and suppose that  $\sigma$  generates the Galois group of  $L$  over  $K$ . Let  $m$  be an integer satisfying  $1 \leq m \leq n$  and let  $w(t)$  be a polynomial of degree  $m$  in  $L[t]$ . Let*

$$R = \{x \in L : w(\sigma)x = 0\}.$$

*Then we have  $\dim_K R \leq m$ .*

It is well known that Frobenius automorphism  $\sigma(x) = x^2$  generates the cyclic group  $\text{Gal}(\mathbb{F}_{2^k}/\mathbb{F}_2) \simeq \mathbb{Z}/k\mathbb{Z}$ . Moreover, if  $\gcd(i, k) = 1$ , then  $\sigma^i(x) = x^{2^i}$  is also a generators. We have the following corollary.

**Corollary 3.** *Suppose  $k$  is an integer and  $\gcd(i, k) = 1$ . For any  $c_1, c_2, c_3 \in \mathbb{F}_{2^k}$  with not all zero, then the following equation*

$$c_1 x^{2^{2i}} + c_2 x^{2^i} + c_3 x = 0$$

*has at most 4 solutions over  $\mathbb{F}_{2^k}$ .*

Moreover, if  $k$  is an odd integer and  $\gcd(i, k) = 1$ , then  $\gcd(2i, k) = 1$ . The next corollary is obviously.

**Corollary 4.** *Suppose  $k$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $c_1, c_2, c_3 \in \mathbb{F}_{2^k}$  with not all zero, then the following equation*

$$c_1 x^{2^{4i}} + c_2 x^{2^{2i}} + c_3 x = 0$$

*has at most 4 solutions over  $\mathbb{F}_{2^k}$ .*

### 3. Butterfly Structure with $\alpha \neq 0, 1$

When  $k$  is odd,  $\gcd(i, k) = 1$ , we have also  $\gcd(2i, k) = 1$ , which implies that  $\gcd(2^i \pm 1, 2^k - 1) = 1$ . Both maps  $x \mapsto x^{2^i+1}$  and  $x \mapsto x^{2^i-1}$  are bijective over  $\mathbb{F}_{2^k}$ .

In this section, we study the butterfly structures with  $\alpha \neq 0, 1$  for block sizes  $2k$  ( $k$  odd). In section 3.1, we show that these structures are always differential 4-uniform. In section 3.2, the algebraic degree is given. In section 3.3, we show that the nonlinearity of these structures are optimal.

### 3.1. Differential uniformity

In order to characterize the differential uniformity, we need the following lemma firstly.

**Lemma 5.** *Let  $k$  is an odd integer and  $\gcd(i, k) = 1$ . Then for any  $\alpha \in \mathbb{F}_{2^k}$  with  $\alpha \neq 0, 1$ , the following system of equations in variables  $u, v$*

$$\begin{cases} (\alpha v + u) \left( \alpha(\alpha u + v)^{2^i} + u^{2^i} \right) + (\alpha v + u)^{2^i} \left( \alpha^{2^i}(\alpha u + v) + u \right) = 0, & (1) \\ (\alpha v + u)(\alpha u + v) + \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \alpha^{2^i}(\alpha v + u) + v \right) = 0, & (2) \\ (\alpha v + u)(\alpha u + v)^{2^i} + \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \alpha(\alpha v + u)^{2^i} + v^{2^i} \right) = 0 & (3) \end{cases}$$

holds over  $\mathbb{F}_{2^k}$  if and only if  $u, v$  satisfying  $\alpha v + u = 0$  and  $\alpha^{2^i}(\alpha u + v) + u = 0$ .

PROOF. The sufficiency is obvious. Now we show the necessary.

If  $\alpha v + u = 0$ ,  $\alpha^{2^i}(\alpha u + v) + u \neq 0$ , then  $u \neq 0, v \neq 0$ . Eq.(2) becomes  $\left( \alpha^{2^i}(\alpha u + v) + u \right) v = 0$ , which contradicts that  $\alpha^{2^i}(\alpha u + v) + u \neq 0$  and  $v \neq 0$ .

If  $\alpha v + u \neq 0$ ,  $\alpha^{2^i}(\alpha u + v) + u = 0$ , then from Eq.(2) we get  $\alpha u + v = 0$ . Note that  $\alpha^{2^i}(\alpha u + v) + u = 0$ , we have  $u = v = 0$ , which contradicts that  $\alpha v + u \neq 0$ .

Now we always assume that  $\alpha v + u \neq 0$ ,  $\alpha^{2^i}(\alpha u + v) + u \neq 0$ .

If  $u = 0$ , then  $v \neq 0$ . According to Eq.(1), We obtain  $(\alpha^{2^i} + \alpha)^2 = 0$ , which is impossible since  $\alpha \neq 0, 1$  and  $\gcd(i, k) = 1$ .

If  $v = 0$ , then  $u \neq 0$ . From Eq.(2) and Eq.(3), we have  $\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha = 0$  and  $\alpha^{2^i+2} + \alpha^{2^i} + \alpha = 0$ . Hence  $\alpha^{2^{i+1}+1} + \alpha^{2^i+2} = \alpha^{2^i+1}(\alpha^{2^i} + \alpha) = 0$ , which is impossible as well.

We also assume that  $u \neq 0, v \neq 0$ .

Then  $\alpha(\alpha u + v)^{2^i} + u^{2^i} \neq 0$ , otherwise, from Eq.(1) we have  $\alpha v + u = 0$  or  $\alpha^{2^i}(\alpha u + v) + u = 0$ , which contradicts the assumption. And also  $\alpha u + v \neq 0$ , otherwise, from Eq.(3) we have  $u = 0$  or  $\alpha^{2^i}(\alpha u + v) + u = 0$ , which also contradicts the assumption.

According to Eq.(1) and Eq.(3), we have

$$\frac{(\alpha v + u)^{2^i}}{\alpha(\alpha u + v)^{2^i} + u^{2^i}} = \frac{\alpha v + u}{\alpha^{2^i}(\alpha u + v) + u} = \frac{\alpha(\alpha v + u)^{2^i} + v^{2^i}}{(\alpha u + v)^{2^i}} \quad (4)$$

To simplify expression, we denote  $\alpha = \beta^{2^i}$ , then  $\beta \neq 0, 1$ . From Eq.(4) we get

$$\frac{(\alpha v + u)^{2^i}}{(\beta(\alpha u + v) + u)^{2^i}} = \frac{(\beta(\alpha v + u) + v)^{2^i}}{(\alpha u + v)^{2^i}},$$

which is equivalent to

$$(\alpha v + u)(\alpha u + v) = (\beta(\alpha u + v) + u)(\beta(\alpha v + u) + v). \quad (5)$$

We combine the above equation with Eq.(2) and get

$$(\beta(\alpha u + v) + u)(\beta(\alpha v + u) + v) = \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \alpha^{2^i}(\alpha v + u) + v \right),$$

which can be simplify to that

$$(\alpha^{2^i} + \beta)^2(\alpha u + v)(\alpha v + u) + (\alpha^{2^i} + \beta)v(\alpha u + v) + (\alpha^{2^i} + \beta)u(\alpha v + u) = 0.$$

Notice that  $\alpha^{2^i} + \beta = \beta^{2^{2i}} + \beta \neq 0$  since  $\beta \neq 0, 1$ ,  $k$  odd and  $\gcd(i, k) = 1$ , dividing by  $\alpha^{2^i} + \beta$  we get

$$(\alpha^{2^i} + \beta)(\alpha u + v)(\alpha v + u) + v(\alpha u + v) + u(\alpha v + u) = 0.$$

Denote  $u = \lambda v$ , then  $\lambda \neq 0$ , the above equation becomes

$$(\alpha^{2^{i+1}} + \alpha\beta + 1)\lambda^2 + (\alpha^{2^i} + \beta)(\alpha^2 + 1)\lambda + (\alpha^{2^{i+1}} + \alpha\beta + 1) = 0. \quad (6)$$

In case of  $\alpha^{2^{i+1}} + \alpha\beta + 1 = 0$ . Note that  $(\alpha^{2^i} + \beta)(\alpha^2 + 1) \neq 0$ , then  $\lambda = 0$ , which is a contradiction. Hence  $\alpha^{2^{i+1}} + \alpha\beta + 1 \neq 0$ . By dividing  $\alpha^{2^{i+1}} + \alpha\beta + 1$ , Eq.(6) is equivalent to

$$\lambda^2 + \frac{(\alpha^{2^i} + \beta)(\alpha^2 + 1)}{\alpha^{2^{i+1}} + \alpha\beta + 1}\lambda + 1 = 0 \quad (7)$$

We replace Eq.(2) by  $u = \lambda v$  and get

$$(\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha)\lambda^2 + (\alpha^{2^{i+1}+2} + \alpha^{2^{i+1}} + \alpha^2)\lambda + (\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha) = 0$$

Similarly, if  $\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha = 0$ , since  $\alpha^{2^{i+1}+2} + \alpha^{2^{i+1}} + \alpha^2 = (\alpha + 1)^{2(2^i+1)} + 1 \neq 0$ , we obtain  $\lambda = 0$ , which is a contradiction. By dividing  $\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha$ , the above equation becomes

$$\lambda^2 + \frac{\alpha^{2^{i+1}+2} + \alpha^{2^{i+1}} + \alpha^2}{\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha}\lambda + 1 = 0. \quad (8)$$

For each equation in variable  $\lambda$  of Eq.(7) and Eq.(8), either has two solutions or has no solution over  $\mathbb{F}_{2^k}$ . Moreover, the two solutions are the inverse of each other. It is readily to verify that the intersection of solutions sets of two equations is identical or empty. If the intersection is empty, then at least one of Eq.(1)-(3) does not hold. If the two solutions sets is identical, we must have

$$\frac{(\alpha^{2^i} + \beta)(\alpha^2 + 1)}{\alpha^{2^{i+1}} + \alpha\beta + 1} = \frac{\alpha^{2^{i+1}+2} + \alpha^{2^{i+1}} + \alpha^2}{\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha}.$$

Replace  $\alpha = \beta^{2^i}$ , we deduce that

$$\beta^{2^{2i}+2^{i+1}+1} + \beta^{2^{2i}+1} + \beta^{2^{2i}+2^i} + \beta^{2^{i+1}} + \beta^{2^i+1} = 0,$$

which, furthermore, is equal to the following equation

$$\beta^{(2^i+1)^2} = (\beta^{2^i} + \beta)^{2^i+1}.$$

Notice that  $x^{2^i+1}$  is a permutation over  $\mathbb{F}_{2^k}$ , hence we have

$$\beta^{2^i+1} + \beta^{2^i} + \beta = (\beta + 1)^{2^i+1} + 1 = 0$$

which is contradict to that  $\beta \neq 0$ . Hence, at least one of the Eq.(1)-(3) does not hold.

We complete the proof.  $\square$

**Theorem 6.** *Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $0 \leq t \leq k - 1$ ,  $\alpha \in \mathbb{F}_{2^k}$ , and  $\alpha \neq 0, 1$ , let  $H_e^\alpha$  and  $V_e^\alpha$  be the open and closed  $2k$ -bit butterflies structure with exponent  $e = (2^i + 1) \times 2^t$  and coefficient  $\alpha$ . Then the differential uniformity of both  $H_e^\alpha$  and  $V_e^\alpha$  is at most equal to 4.*

PROOF. As  $H_e^\alpha$  and  $V_e^\alpha$  are CCZ-equivalent, they have the same differential uniformity. It is sufficient to prove that the differential uniformity of  $V_e^\alpha$  is 4. Besides, the functions  $V_e^\alpha$  with exponent  $e = (2^i + 1) \times 2^t$  is affine equivalent to functions  $V_e^\alpha$  with exponent  $e = 2^i + 1$ . Thus it is sufficient to consider the case where the exponent is equal to  $e = 2^i + 1$ .

Let  $u, v, a, b \in \mathbb{F}_{2^k}$  and  $(u, v) \neq (0, 0)$ . Then we need to prove that

$$V_e^\alpha(x, y) + V_e^\alpha(x + u, y + v) = (a, b), \quad (9)$$

namely, the following system of equations

$$\begin{cases} \left( \alpha^{2^i}(\alpha u + v) + u \right) x^{2^i} + \left( \alpha(\alpha u + v)^{2^i} + u^{2^i} \right) x \\ \quad + (\alpha u + v)y^{2^i} + (\alpha u + v)^{2^i}y = (\alpha u + v)^{2^i+1} + u^{2^i+1} + a, \\ \left( \alpha v + u \right) x^{2^i} + (\alpha v + u)^{2^i}x \\ \quad + \left( \alpha^{2^i}(\alpha v + u) + v \right) y^{2^i} + \left( \alpha(\alpha v + u)^{2^i} + v^{2^i} \right) y = (\alpha v + u)^{2^i+1} + v^{2^i+1} + b. \end{cases}$$

has at most 4 solutions over  $\mathbb{F}_{2^k}^2$ , which is equivalent to the linear homogeneous system of above equations

$$\begin{cases} \left( \alpha^{2^i}(\alpha u + v) + u \right) x^{2^i} + \left( \alpha(\alpha u + v)^{2^i} + u^{2^i} \right) x + (\alpha u + v)y^{2^i} + (\alpha u + v)^{2^i}y = 0, & (10) \\ \left( \alpha v + u \right) x^{2^i} + (\alpha v + u)^{2^i}x + \left( \alpha^{2^i}(\alpha v + u) + v \right) y^{2^i} + \left( \alpha(\alpha v + u)^{2^i} + v^{2^i} \right) y = 0, & (11) \end{cases}$$

has at most 4 solutions over  $\mathbb{F}_{2^k}^2$ .

(I) In the case of  $\alpha^{2^i}(\alpha u + v) + u = 0$ . Then  $\alpha u + v \neq 0$ , otherwise, we have  $u = v = 0$ , which is impossible. And also  $\alpha(\alpha u + v)^{2^i} + u^{2^i} = \alpha(\alpha u + v)^{2^i} + \left( \alpha^{2^i}(\alpha u + v) \right)^{2^i} = (\alpha^{2^{2i}} + \alpha)(\alpha u + v)^{2^i} \neq 0$  since  $\alpha \neq 0, 1$ ,  $k$  odd and  $\gcd(i, k) = 1$ . Then Eq.(10) can be written

$$x = \frac{\alpha u + v}{(\alpha^{2^{2i}} + \alpha)(\alpha u + v)^{2^i}} y^{2^i} + \frac{1}{\alpha^{2^{2i}} + \alpha} y. \quad (12)$$

If  $\alpha v + u = 0$ , Eq.(11) becomes  $vy^{2^i} + v^{2^i}y = 0$ . Note that we must have  $v \neq 0$ , therefore,  $y = 0$  or  $y = v$ . Eq.(12) in  $x$  have only one solution with respect to each of  $y$ . Hence, the total number of solutions is equal to 2.

If  $\alpha v + u \neq 0$ . We replace Eq.(11) by Eq.(12) and get

$$A_1 y^{2^{2i}} + A_2 y^{2^i} + A_3 y = 0,$$

where

$$\begin{aligned} A_1 &= \frac{(\alpha v + u)(\alpha u + v)^{2^i}}{(\alpha^{2^{2i}} + \alpha)^{2^i}(\alpha u + v)^{2^{2i}}}, \\ A_2 &= \frac{(\alpha v + u)(\alpha u + v)^{2^i}}{(\alpha^{2^{2i}} + \alpha)^{2^i}(\alpha u + v)^{2^{2i}}} + \frac{(\alpha v + u)^{2^i}(\alpha u + v)}{(\alpha^{2^{2i}} + \alpha)^{2^i}(\alpha u + v)^{2^{2i}}} + \alpha^{2^i}(\alpha v + u) + v, \\ A_3 &= \frac{(\alpha v + u)^{2^i}(\alpha u + v)^{2^i}}{(\alpha^{2^{2i}} + \alpha)(\alpha u + v)^{2^i}} + \alpha(\alpha v + u)^{2^i} + v^{2^i}. \end{aligned}$$

Notice that  $A_1 \neq 0$ , hence, according to Corollary 3, the above equation in  $y$  has at most 4 solutions. From Eq.(12), the solution  $x$  is uniquely determined by  $y$ . Hence, the total number of solutions is at most equal to 4.

(II) In the case of  $\alpha^{2^i}(\alpha u + v) + u \neq 0$ .

If  $\alpha v + u = 0$ , Eq.(11) becomes  $vy^{2^i} + v^{2^i}y = 0$ . Note that we must have  $v \neq 0$ , therefore,  $y = 0$  or  $y = v$ . For each of the solutions  $y$ , Eq.(10) in  $x$  has at most 2 solutions. Hence, the total number of solutions is at most equal to 4.

If  $\alpha v + u \neq 0$ . We add Eq.(10) multiplied by  $\alpha v + u$  to Eq.(11) multiplied by  $\alpha^{2^i}(\alpha u + v) + u$ , from which we eliminate  $x^{2^i}$  and get

$$A_4x + A_5y^{2^i} + A_6y = 0, \quad (13)$$

where

$$\begin{aligned} A_4 &= (\alpha v + u) \left( \alpha(\alpha u + v)^{2^i} + u^{2^i} \right) + (\alpha v + u)^{2^i} \left( \alpha^{2^i}(\alpha u + v) + u \right), \\ A_5 &= (\alpha v + u)(\alpha u + v) + \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \alpha^{2^i}(\alpha v + u) + v \right), \\ A_6 &= (\alpha v + u)(\alpha u + v)^{2^i} + \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \alpha(\alpha v + u)^{2^i} + v^{2^i} \right). \end{aligned}$$

According to Lemma 5, not all of  $A_4, A_5, A_6$  are equal to 0.

If  $A_4 = 0$ , from Eq.(13),  $y$  has at most 2 solutions. For each of the solutions  $y$ , Eq.(10) in  $x$  has at most 2 solutions. Hence, the total number of solutions is at most equal to 4.

If  $A_4 \neq 0$  and  $A_5 = A_6 = 0$ , then  $x = 0$ . Since not both of  $\alpha^{2^i}(\alpha v + u) + v$  and  $\alpha(\alpha v + u)^{2^i} + v^{2^i}$  equal to 0, otherwise, we get  $(\alpha^{2^{2i}} + \alpha)(\alpha v + u)^{2^i} = 0$ , which is impossible. Replace Eq.(11) by  $x = 0$ , we obtain a equation in  $y$  with coefficient not all zero with at most 2 solutions. Hence, the total number of solutions is at most equal to 2.

If  $A_4 \neq 0, A_5 \neq 0, A_6 \neq 0$ , we replace Eq.(10) by Eq.(13) and get

$$A_7y^{2^i} + A_8y = 0, \quad (14)$$

where

$$\begin{aligned} A_7 &= \left( \alpha^{2^i}(\alpha u + v) + u \right) \left( \frac{A_6}{A_4} \right)^{2^i} + (\alpha u + v), \\ A_8 &= \left( \alpha(\alpha u + v)^{2^i} + u^{2^i} \right) \frac{A_6}{A_4} + (\alpha u + v)^{2^i}. \end{aligned}$$



We claim that  $A_8 \neq 0$ , otherwise,

$$\left(\alpha(\alpha u + v)^{2^i} + u^{2^i}\right) A_6 = (\alpha u + v)^{2^i} A_4.$$

Replace above equation by the expressions  $A_4$  and  $A_6$ , recall that  $\alpha^{2^i}(\alpha u + v) + u \neq 0$ , and we get

$$(\alpha v + u)(\alpha u + v) = (\beta(\alpha u + v) + u)(\beta(\alpha v + u) + v), \quad (15)$$

where  $\alpha = \beta^{2^i}$ . If  $u = 0$ , then  $v \neq 0$ . From  $A_5 = 0$  and Eq.(15), we obtain  $\beta^{2^i+2} + \beta^{2^i} + \beta = 0$  and  $\alpha^{2^{i+1}+1} + \alpha^{2^i} + \alpha = 0$ . We add the first equation raised up to  $2^i$  to the second equation and get  $\alpha^{2^{i+1}}(\alpha^{2^i} + \alpha) = 0$ , which is impossible. The case  $v = 0, u \neq 0$  is identical. Hence, we can assume the  $u \neq 0, v \neq 0$ . Observe that equations  $A_5 = 0$  and Eq.(15) are the same with Eq.(2) and Eq.(5), then according the proof of Lemma 5, this is impossible, which means  $A_8 \neq 0$ . Therefore, Eq.(14) in  $y$  has at most 2 solutions. For each of the solutions, Eq.(13) in  $x$  has only one solution. Hence, Eq.(9) has at most 4 solutions.

If  $A_4 \neq 0$  and  $A_5 \neq 0$ , we replace Eq.(11) by Eq.(13) and get

$$A_9 y^{2^{2i}} + A_{10} y^{2^i} + A_{11} y = 0, \quad (16)$$

where  $A_9 = (\alpha v + u)(\frac{A_5}{A_4})^{2^i} \neq 0$  and  $A_{10}, A_{11}$  are expressions of  $\alpha, u, v$ . According to Corollary 3, Eq.(16) in  $y$  has at most 4 solutions. For each solution, Eq.(13) in  $x$  has only one solution. Hence, Eq.(9) has at most 4 solutions.

Therefore, Eq.(9) has at most 4 solutions, meaning that the differential uniformity of  $V_e^\alpha$  is at most 4. We complete the proof.  $\square$

**Remark 1.** In [9], it is proved that the 6-bit APN permutation described by Dillon et al. is affine equivalent to the butterfly structures  $H_6^2$ . However, when  $k > 3$ , the pair  $(e, \alpha)$  such that  $H_e^\alpha$  is APN has not been found. The authors verified experimentally that butterfly structures are never differentially 4-uniform for  $k = 4, 8, 10$ , while cases  $k = 6$  there does exist.

### 3.2. Algebraic Degree

When  $e = 2^i + 1$ , the left and right side of  $V_e^\alpha(x, y)$  are equal to  $(\alpha x + y)^{2^i+1} + x^{2^i+1}$ ,  $(x + \alpha y)^{2^i+1} + y^{2^i+1}$  respectively. It is obvious that it is quadratic. Now we consider the open butterfly structure  $H_e^\alpha$ . The following result is needed.

**Lemma 7 ([6]).** Suppose  $k$  is odd integer and  $\gcd(i, k) = 1$ . Then the compositional inverse of  $x^{2^i+1}$  over  $\mathbb{F}_{2^k}$  is  $x^t$ , where  $t = \sum_{j=0}^{\frac{k-1}{2}} 2^{2ji} \pmod{(2^k - 1)}$ . Its algebraic degree is  $\frac{k+1}{2}$ .

The right side of the output of the open butterfly  $H_e^\alpha$  is equal to  $(x + y^{2^i+1})^{\frac{1}{2^i+1}} + \alpha y$ . According to Lemma 7, we deduce from this expression that  $(x + y^{2^i+1})^{\frac{1}{2^i+1}}$  is equal to  $\prod_{j=0}^{\frac{k-1}{2}} (x + y^{2^i+1})^{2^{2ji}}$ . This sum can be developed as follows:

$$(x + y^{2^i+1})^{\frac{1}{2^i+1}} = \sum_{J \subseteq [0, (k-1)/2]} \underbrace{\prod_{j \in J} y^{(2^i+1) \cdot 2^{2ji}}}_{\deg \leq 2|J|} \underbrace{\prod_{j \in \bar{J}} x^{2^{2ji}}}_{\deg = (k+1)/2 - |J|},$$

where  $\bar{J}$  is the complement of  $J$  in  $[0, (k-1)/2]$ . The algebraic degree of each term in this sum is at most equal to  $(k+1)/2 + |J|$ . If  $\bar{J} = \emptyset$  then  $x$  is absent from the term so that the algebraic degree of this term is equal to  $w_2\left(\sum_{j=0}^{(k-1)/2} (2^i+1) \cdot 2^{2ji}\right) = w_2(1) = 1$ . If  $|\bar{J}| = 1$ , then the algebraic degree of this term is at most  $(k-1)/2 + (k+1)/2 = k$ . Moreover, if  $\bar{J} = \{0\}$ , then the term is equal to  $x(y^{2^i})^{-1}$  which has algebraic degree  $1 + (k-1) = k$ . If  $|\bar{J}| > 1$ , the degree of these terms is at most equal to  $(k-3)/2 + (k+1)/2 = k-1$ . Thus, the right side of the output has an algebraic degree equal to  $k$ .

The left side is equal to

$$\left(y + \alpha \left( (x + y^{2^i+1})^{\frac{1}{2^i+1}} + \alpha y \right)\right)^{2^i+1} + \left( (x + y^{2^i+1})^{\frac{1}{2^i+1}} + \alpha y \right)^{2^i+1}.$$

The terms of highest algebraic degree in this equation are of the shape  $y^{2^i} (x + y^{2^i+1})^{\frac{1}{2^i+1}}$  and  $y(x + y^{2^i+1})^{2^i \cdot \frac{1}{2^i+1}}$ . We still have

$$y^{2^i} (x + y^{2^i+1})^{\frac{1}{2^i+1}} = \sum_{J \subseteq [0, (k-1)/2]} \underbrace{y^{2^i} \cdot \prod_{j \in J} y^{(2^i+1) \cdot 2^{2ji}}}_{\deg \leq 2|J|+1} \underbrace{\prod_{j \in \bar{J}} x^{2^{2ji}}}_{\deg = (k+1)/2 - |J|},$$

so that the algebraic degree of each term is at most equal to  $(k+1)/2 + |J| + 1$ . If  $|J| = (k+1)/2$ , i.e.,  $\bar{J} = \emptyset$ , then  $x$  is absent from the term so that the algebraic degree of this term is equal to  $w_2\left(2^i + \sum_{j=0}^{(k-1)/2} (2^i+1) \cdot 2^{2ji}\right) = w_2(2^i+1) = 2$ . If  $|J| \leq (k-1)/2$ , then the algebraic degree of these terms is at most equal to  $(k+1)/2 + |J| + 1 \leq k+1$ . For  $J = [1, (k-1)/2]$ , then the algebraic degree of this term is  $1 + w_2\left(2^i + (2^i+1) \sum_{j=1}^{(k-1)/2} 2^{2ji}\right) = 1 + w_2\left(\sum_{j=0}^{(k-1)/2} 2^{2ji}\right) = k+1$ . The terms  $y(x + y^{2^i+1})^{2^i \cdot \frac{1}{2^i+1}}$  are treated similarly. Hence the left side of the output has algebraic degree  $k+1$ .

**Theorem 8.** *Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $0 \leq t \leq k-1$ ,  $\alpha \in \mathbb{F}_{2^k}$ , and  $\alpha \neq 0, 1$ , let  $H_e^\alpha$  and  $V_e^\alpha$  be the open and closed  $2k$ -bit butterflies structure with exponent  $e = (2^i+1) \times 2^t$  and coefficient  $\alpha$ . Then*

- (1) *The algebraic degree of  $V_e^\alpha$  is 2.*
- (2) *The left and right side of the output of  $H_e^\alpha$  have algebraic degree  $k+1$  and  $k$  respectively.*

### 3.3. Nonlinearity

In this section, we consider the nonlinearity, the following lemma is needed.

**Lemma 9.** *Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . Then for any  $\alpha \in \mathbb{F}_{2^k}$  with  $\alpha \neq 0, 1$ , the following system of equations in variables  $c, d$*

$$\begin{cases} \left( \alpha^{2^i+1}c + c + d \right) \left( \alpha^{2^i}c + \alpha d \right)^{2^i} + \left( \alpha^{2^i+1}c + c + d \right)^{2^i} \left( \alpha c + \alpha^{2^i}d \right) = 0, & (17) \\ \left( \alpha c + \alpha^{2^i}d \right)^{2^i} \left( \alpha^{2^i}c + \alpha d \right)^{2^i} + \left( \alpha^{2^i+1}c + c + d \right)^{2^i} \left( \alpha^{2^i+1}d + c + d \right)^{2^i} = 0, & (18) \\ \left( \alpha^{2^i}c + \alpha d \right) \left( \alpha^{2^i}c + \alpha d \right)^{2^i} + \left( \alpha^{2^i+1}c + c + d \right)^{2^i} \left( \alpha^{2^i+1}d + c + d \right) = 0 & (19) \end{cases}$$

holds over  $\mathbb{F}_{2^k}$  if and only if  $c, d$  satisfying  $\alpha^{2^i+1}c + c + d = 0$  and  $\alpha^{2^i}c + \alpha d = 0$ .

PROOF. The sufficiency is obvious. Now we show the necessary.

If  $\alpha^{2^i+1}c + c + d = 0, \alpha^{2^i}c + \alpha d \neq 0$ , which contradicts Eq.(19). If  $\alpha^{2^i+1}c + c + d \neq 0, \alpha^{2^i}c + \alpha d = 0$ , from Eq.(17) we must have  $\alpha c + \alpha^{2^i}d = 0$ , which implies that  $(\alpha^{2^i} + \alpha)(c + d) = 0$ . Hence,  $c = d = 0$ , which contradicts  $\alpha^{2^i+1}c + c + d \neq 0$ .

Now we always assume that  $\alpha^{2^i+1}c + c + d \neq 0, \alpha^{2^i}c + \alpha d \neq 0$ . Then  $\alpha c + \alpha^{2^i}d \neq 0$  and  $\alpha^{2^i+1}d + c + d \neq 0$ , otherwise, we must have  $\alpha^{2^i+1}c + c + d = 0$  or  $\alpha^{2^i}c + \alpha d = 0$  from Eq.(17) and Eq.(19).

If  $c = 0$ , and replace it into Eq.(18), we obtain  $d^{2^{i+1}} = 0$ , which contradicts the hypothesis. The case  $d = 0$  is identical. Therefore, we also assume that  $c \neq 0, d \neq 0$ .

According to Eq.(17) and Eq.(18), we have

$$\frac{\alpha^{2^i+1}c + c + d}{\alpha c + \alpha^{2^i}d} = \frac{(\alpha^{2^i+1}c + c + d)^{2^i}}{(\alpha^{2^i}c + \alpha d)^{2^i}} = \frac{(\alpha c + \alpha^{2^i}d)^{2^i}}{(\alpha^{2^i+1}d + c + d)^{2^i}},$$

and obtain the following equation

$$(\alpha^{2^i+1}c + c + d)(\alpha^{2^i+1}d + c + d)^{2^i} = (\alpha c + \alpha^{2^i}d)(\alpha c + \alpha^{2^i}d)^{2^i}.$$

To simplify expressions, we denote  $c = \lambda d, \lambda \neq 0$ , and use the notation  $\beta = \alpha^{2^i}, \beta \neq 0, 1$ . The above equation can be rewritten as

$$\lambda^{2^i+1} + (\beta^2 + 1)\lambda^{2^i} + (\alpha\beta^{2^i+2} + \beta^{2^i+1} + \alpha\beta^{2^i} + \alpha\beta + 1)\lambda + 1 = 0. \quad (20)$$

Similarly, Eq.(19) can be rewritten as

$$\lambda^{2^i+1} + (\alpha\beta^{2^i+2} + \beta^{2^i+1} + \alpha\beta^{2^i} + \alpha\beta + 1)\lambda^{2^i} + (\beta^2 + 1)\lambda + 1 = 0. \quad (21)$$

We add Eq.(20) to Eq.(21) and get

$$(\alpha\beta^{2^i+2} + \beta^{2^i+1} + \alpha\beta^{2^i} + \alpha\beta + \beta^2)\lambda^{2^i} + (\alpha\beta^{2^i+2} + \beta^{2^i+1} + \alpha\beta^{2^i} + \alpha\beta + \beta^2)\lambda = 0. \quad (22)$$

Recall that  $\alpha\beta^{2^i+2} + \beta^{2^i+1} + \alpha\beta^{2^i} + \alpha\beta + \beta^2 = \alpha^{2^{2i}+2^{i+1}+1} + \alpha^{2^{2i}+2^i} + \alpha^{2^{2i}+1} + \alpha^{2^{i+1}} + \alpha^{2^i+1} \neq 0$ , then Eq.(22) becomes  $\lambda^{2^i} + \lambda = 0$ . Hence,  $\lambda = 1$ , which means  $c = d$ . Replace Eq.(18) by  $c = d$  and we have  $((\alpha + 1)^{2^i+1} + 1)^2 c^2 = 0$ , which is a contradiction since  $\alpha \neq 0$  and  $c \neq 0$ .

We complete the proof.  $\square$

**Lemma 10.** Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . Then for any  $(c, d) \in \mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$  with  $(c, d) \neq (0, 0)$ , the following system of equations in variables  $u, v$

$$\begin{cases} (\alpha^{2^i+1}c + c + d)^{2^i} u^{2^{2i}} + (\alpha^{2^i+1}c + c + d)u + (\alpha c + \alpha^{2^i}d)^{2^i} v^{2^{2i}} + (\alpha^{2^i}c + \alpha d)v = 0, & (23) \\ (\alpha^{2^i}c + \alpha d)^{2^i} u^{2^{2i}} + (\alpha c + \alpha^{2^i}d)u + (\alpha^{2^i+1}d + c + d)^{2^i} v^{2^{2i}} + (\alpha^{2^i+1}d + c + d)v = 0 & (24) \end{cases}$$

has at most 4 solutions over  $\mathbb{F}_{2^k}^2$ .

PROOF. (I) In the case of  $\alpha^{2^i+1}c + c + d = 0$ , Eq.(23) becomes

$$\left(\alpha c + \alpha^{2^i}d\right)^{2^i} v^{2^{2i}} + \left(\alpha^{2^i}c + \alpha c\right)v = 0. \quad (25)$$

If  $\alpha c + \alpha^{2^i}d = 0$ , then  $\alpha^{2^i}c + \alpha d \neq 0$ , otherwise, we can obtain  $c = d = 0$ . Hence,  $v = 0$ , replace it into Eq.(24) and we get  $u = 0$ . So we have only solution  $(0, 0)$ .

If  $\alpha^{2^i}c + \alpha d = 0$ , then  $\alpha c + \alpha^{2^i}d \neq 0$ . Similarly, we have only solution  $(0, 0)$ .

If  $\alpha^{2^i}c + \alpha d \neq 0, \alpha c + \alpha^{2^i}d \neq 0$ , then Eq.(25) in  $v$  has 2 solutions. For each one  $v$  of the solutions, Eq.(24) in  $x$  has at most 2 solutions. Hence, the total numbers of solutions is at most equal to 4.

(II) In the case of  $\alpha^{2^i+1}c + c + d \neq 0$ .

If  $\alpha^{2^i}c + \alpha d = 0$ , then  $\alpha c + \alpha^{2^i}d \neq 0$ . Eq.(24) becomes

$$\left(\alpha c + \alpha^{2^i}d\right)u + \left(\alpha^{2^i+1}d + c + d\right)^{2^i} v^{2^{2i}} + \left(\alpha^{2^i+1}d + c + d\right)v = 0. \quad (26)$$

When  $\alpha^{2^i+1}d + c + d = 0$ , then  $u = 0$ . Replace it into Eq.(23) and we have only one solution  $(0, 0)$ . When  $\alpha^{2^i+1}d + c + d \neq 0$ , Replace Eq.(23) by Eq.(26) and we obtain

$$B_1 v^{2^{4i}} + B_2 v^{2^{2i}} + B_3 v = 0,$$

where  $B_1 = \frac{(\alpha^{2^i+1}c + c + d)(\alpha^{2^i+1}d + c + d)^{2^{3i}}}{(\alpha c + \alpha^{2^i}d)^{2^{2i}}} \neq 0$ ,  $B_2, B_3$  are expressions of  $\alpha, c, d$ . According to

Corollary 4, the above equation in  $v$  has at most 4 solutions. For each solution, Eq.(26) in  $u$  has only one solution. Hence, the total numbers of solutions is at most equal to 4.

If  $\alpha^{2^i}c + \alpha d \neq 0$ , we add Eq.(23) multiplied by  $\left(\alpha^{2^i}c + \alpha d\right)^{2^i}$  to Eq.(24) multiplied by  $\left(\alpha^{2^i+1}c + c + d\right)^{2^i}$  to eliminate  $u^{2^{2i}}$  and get

$$B_4 u + B_5 v^{2^{2i}} + B_6 v = 0, \quad (27)$$

where

$$\begin{aligned} B_4 &= \left(\alpha^{2^i+1}c + c + d\right) \left(\alpha^{2^i}c + \alpha d\right)^{2^i} + \left(\alpha^{2^i+1}c + c + d\right)^{2^i} \left(\alpha c + \alpha^{2^i}d\right), \\ B_5 &= \left(\alpha c + \alpha^{2^i}d\right)^{2^i} \left(\alpha^{2^i}c + \alpha d\right)^{2^i} + \left(\alpha^{2^i+1}c + c + d\right)^{2^i} \left(\alpha^{2^i+1}d + c + d\right)^{2^i}, \\ B_6 &= \left(\alpha^{2^i}c + \alpha d\right) \left(\alpha^{2^i}c + \alpha d\right)^{2^i} + \left(\alpha^{2^i+1}c + c + d\right)^{2^i} \left(\alpha^{2^i+1}d + c + d\right). \end{aligned}$$

According to Lemma 9, not all of  $B_4, B_5, B_6$  are equal to 0.

If  $B_4 = 0$ , from Eq.(27),  $v$  has at most 2 solutions. For each of the solutions  $v$ , Eq.(23) in  $u$  has at most 2 solutions. Hence, the total number of solutions is at most equal to 4.

If  $B_4 \neq 0$  and  $B_5 = B_6 = 0$ , then  $u = 0$ . Recall that not both of  $\alpha^{2^i}c + \alpha c$  and  $\alpha c + \alpha^{2^i}d$  are equal to 0, Replace Eq.(23) by  $u = 0$ , we obtain a equation in  $v$  with coefficient not all zero with at most 2 solutions. Hence, the total number of solutions is at most equal to 2.

If  $B_4 \neq 0$  and  $B_5 = 0, B_6 \neq 0$ , then replace Eq.(27) into Eq.(23) and obtain

$$B_7 v^{2^{2i}} = 0$$

where  $B_7 = (\alpha^{2^i} c + \alpha d)^{2^i} B_6^{2^{2i}} + (\alpha^{2^i+1} d + c + d)^{2^i} B_4^{2^{2i}}$ . With a tedious verification (See Appendix) we have  $B_7 \neq 0$ . Hence,  $v = 0$ , which implies that  $u = 0$ . We have only one solution.

If  $B_4 \neq 0$  and  $B_5 \neq 0$ , we replace Eq.(24) by Eq.(27) and get

$$B_8 v^{2^{4i}} + B_9 v^{2^{2i}} + B_{10} v = 0, \quad (28)$$

where  $B_8 = (\alpha^{2^i} c + \alpha d)^{2^i} \left(\frac{B_5}{B_4}\right)^{2^{2i}} \neq 0$  and  $B_9, B_{10}$  are expressions of  $\alpha, c, d$ . According to Corollary 4, Eq.(28) in  $v$  has at most 4 solutions. For each solution, Eq.(27) in  $u$  has only one solution. Hence, the total number of solutions is at most equal to 4.

We complete the proof.  $\square$

**Theorem 11.** *Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $0 \leq t \leq k - 1$ ,  $\alpha \in \mathbb{F}_{2^k}$ , and  $\alpha \neq 0, 1$ , let  $H_e^\alpha$  and  $V_e^\alpha$  be the open and closed  $2k$ -bit butterflies structure with exponent  $e = (2^i + 1) \times 2^t$  and coefficient  $\alpha$ . Then the nonlinearity of both  $H_e^\alpha$  and  $V_e^\alpha$  is  $2^{2k-1} - 2^k$ . Furthermore, their Walsh spectrum are  $\{0, \pm 2^k, \pm 2^{k+1}\}$ .*

PROOF. As  $H_e^\alpha$  and  $V_e^\alpha$  are CCZ-equivalent, they have the same nonlinearity and walsh spectrum. It is sufficient to prove that the nonlinearity of  $V_e^\alpha$  is  $2^{2k-1} - 2^k$ . Besides, the functions  $V_e^\alpha$  with exponent  $e = (2^i + 1) \times 2^t$  is affine equivalent to functions  $V_e^\alpha$  with exponent  $e = 2^i + 1$ . Thus it is sufficient to consider the case where the exponent is equal to  $e = 2^i + 1$ .

Let  $a, b, c, d \in \mathbb{F}_{2^k}$ , and  $(c, d) \neq (0, 0)$ . Then we have

$$\begin{aligned} \mathcal{W}_F((a, b), (c, d)) &= \sum_{x, y \in \mathbb{F}_{2^k}} (-1)^{\text{Tr}(c(\alpha x + y)^{2^i+1} + cx^{2^i+1} + d(x + \alpha y)^{2^i+1} + dy^{2^i+1} + ax + by)} \\ &= \sum_{x, y \in \mathbb{F}_{2^k}} (-1)^{f(x, y)}, \end{aligned}$$

where

$$\begin{aligned} f(x, y) = & \text{Tr} \left( (\alpha^{2^i+1} c + c + d) x^{2^i+1} + (\alpha^{2^i+1} d + c + d) y^{2^i+1} \right. \\ & \left. + (\alpha^{2^i} c + \alpha d) x^{2^i} y + (\alpha c + \alpha^{2^i} d) x y^{2^i} + ax + by \right). \end{aligned}$$

Using the fact that  $\text{Tr}(x) = \text{Tr}(x^{2^i})$ , we deduce the following representation

$$\begin{aligned} & f(x, y) + f(x + u, y + v) \\ = & \text{Tr} \left[ \left( (\alpha^{2^i+1} c + c + d)^{2^i} u^{2^{2i}} + (\alpha^{2^i+1} c + c + d) u + (\alpha c + \alpha^{2^i} d)^{2^i} v^{2^{2i}} + (\alpha^{2^i} c + \alpha d) v \right) x^{2^i} \right. \\ & \left. + \left( (\alpha^{2^i} c + \alpha d)^{2^i} u^{2^{2i}} + (\alpha c + \alpha^{2^i} d) u + (\alpha^{2^i+1} d + c + d)^{2^i} v^{2^{2i}} + (\alpha^{2^i+1} d + c + d) v \right) y^{2^i} \right] \\ & + f(u, v), \end{aligned}$$

then it holds that

$$\begin{aligned}
\mathcal{W}_F^2((a, b), (c, d)) &= \sum_{x, y \in \mathbb{F}_{2^k}} (-1)^{f(x, y)} \times \sum_{u, v \in \mathbb{F}_{2^k}} (-1)^{f(x+u, y+v)} \\
&= \sum_{x, y, u, v \in \mathbb{F}_{2^k}} (-1)^{f(x, y) + f(x+u, y+v)} \\
&= 2^{2k} \sum_{u, v \in S(c, d)} (-1)^{f(u, v)},
\end{aligned}$$

where  $R(c, d)$  is the solution set of the following system of equations with variables  $u, v$

$$\begin{cases} \left( \alpha^{2^i+1}c + c + d \right)^{2^i} u^{2^{2i}} + \left( \alpha^{2^i+1}c + c + d \right) u + \left( \alpha c + \alpha^{2^i}d \right)^{2^i} v^{2^{2i}} + \left( \alpha^{2^i}c + \alpha d \right) v = 0, \\ \left( \alpha^{2^i}c + \alpha d \right)^{2^i} u^{2^{2i}} + \left( \alpha c + \alpha^{2^i}d \right) u + \left( \alpha^{2^i+1}d + c + d \right)^{2^i} v^{2^{2i}} + \left( \alpha^{2^i+1}d + c + d \right) v = 0. \end{cases}$$

Denote  $m = \dim_{F_2} R(c, d)$ , according Lemma 10,  $0 \leq m \leq 2$ . Notice that  $f(x, y) + f(x+u, y+v) = f(u, v)$  for  $(u, v) \in R(c, d)$  and  $(x, y) \in \mathbb{F}_{2^k}^2$ , which means  $f(u, v)$  is linear over  $R(c, d)$ . Since  $(0, 0) \in R(c, d)$ , therefore,  $f(u, v)$  is a balanced or constant 0 over  $R(c, d)$ . Thus

$$\mathcal{W}_F^2((a, b), (c, d)) = \begin{cases} 2^{2k+m} & f(u, v) = 0 \text{ over } R(c, d), \\ 0 & \text{otherwise.} \end{cases}$$

As  $\mathcal{W}_F((a, b), (c, d))$  is an integer,  $m$  must be even, i.e.,  $m = 0$  or  $m = 2$ . Hence,  $\mathcal{W}_F((a, b), (c, d)) \in \{0, \pm 2^k, \pm 2^{k+1}\}$ .

Since  $H_e^\alpha$  is a permutation over  $\mathbb{F}_{2^k}^2$ ,  $\mathcal{W}_F((0, 0), (c, d)) = 0$  for any  $(c, d) \in \mathbb{F}_{2^k}^2$  with  $(c, d) \neq (0, 0)$ , which means  $0 \in \Lambda_F$ . Besides, we also have  $\pm 2^{k+1} \in \Lambda_F$ , otherwise, according to Parsevals equality we must have  $\mathcal{W}_F((a, b), (c, d)) = \pm 2^k$  for any  $(a, b), (c, d) \in \mathbb{F}_{2^k}^2$  with  $(c, d) \neq (0, 0)$ , which is impossible. If  $\pm 2^k \notin \Lambda_F$ , according to Lemma 1, the algebraic degree is at most equal to  $2k - (k+1) + 1 = k$ , which contradicts the algebraic degree of  $H_e^\alpha$  is  $k+1$ . Therefore,  $\Lambda_F = \{0, \pm 2^k, \pm 2^{k+1}\}$ , and the nonlinearity  $\mathcal{NL}(F) = 2^{2k-1} - 2^k$ .

We complete the proof.  $\square$

**Remark 2.** Recall that the Walsh spectrum of Gold functions are  $\{0, \pm 2^{k+1}\}$ , which is different from that of butterfly structures. Hence, the butterfly structures  $H_e^\alpha$  and  $V_e^\alpha$  is CCZ-inequivalent to the Gold functions. Besides, in the proof of Lemma 10, there exists some cases that the solution sets  $R(c, d)$  has only one solution  $(0, 0)$  (e.g. the case of  $\alpha c + \alpha^{2^i}d = 0$  and  $\alpha^{2^i}c + \alpha c \neq 0$ ), namely,  $m = 0$ . Hence, we also have  $\pm 2^k \in \Lambda_F$ . From the proof of above theorem, we have actually  $m = 0$  or  $m = 2$ , meaning that the system of equations in Lemma 10 has one solution or 4 solutions.

#### 4. Butterfly Structure with $\alpha = 1$

In this section, we study the butterflies with trivial coefficient  $\alpha = 1$ . We show that  $V_e^1$  is also a permutation in section 4.1. In section 4.2 we consider other cryptographic properties.

#### 4.1. The bijective of closed butterfly structure

When  $\alpha = 1, e = 2^i + 1$ , the closed butterfly  $V_e^1$  becomes

$$V_e^1(x, y) = ((x + y)^{2^i+1} + x^{2^i+1}, (x + y)^{2^i+1} + y^{2^i+1}).$$

Then we have the following result.

**Theorem 12.** *Let  $n$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $0 \leq t \leq k - 1$ , let  $V_e^1$  be the closed  $2k$ -bit butterflies structure with exponent  $e = (2^i + 1) \times 2^t$ . Then  $V_e^1(x, y)$  is a permutation over  $\mathbb{F}_{2^k}^2$ .*

PROOF. Similarly, we consider the case  $e = 2^i + 1$ . For any  $u, v \in \mathbb{F}_{2^k}$  and  $(u, v) \neq (0, 0)$ . It is sufficient to show that

$$V_e^1(x, y) + V_e^1(x + u, y + v) = (0, 0),$$

namely, the system of equations

$$\begin{cases} vx^{2^i} + v^{2^i}x + (u + v)y^{2^i} + (u + v)^{2^i}y = (u + v)^{2^i+1} + u^{2^i+1}, \\ (u + v)x^{2^i} + (u + v)^{2^i}x + uy^{2^i} + u^{2^i}y = (u + v)^{2^i+1} + v^{2^i+1}. \end{cases}$$

has no solution over  $\mathbb{F}_{2^k}^2$ . We consider the following equivalent system of equations

$$\begin{cases} ux^{2^i} + u^{2^i}x + vy^{2^i} + v^{2^i}y = u^{2^i+1} + v^{2^i+1}, \\ (u + v)x^{2^i} + (u + v)^{2^i}x + uy^{2^i} + u^{2^i}y = (u + v)^{2^i+1} + v^{2^i+1}. \end{cases} \quad (29)$$

First, if  $u = 0$ , then  $v \neq 0$ . So equation Eq.(29) becomes

$$vy^{2^i} + v^{2^i}y = v^{2^i+1},$$

which, in fact, is equivalent to  $(v + y)^{2^i+1} = y^{2^i+1}$ . Therefore, equation (29) has no solution over  $\mathbb{F}_{2^k}^2$  since  $x^{2^i+1}$  is a permutation over  $\mathbb{F}_{2^k}$ .

The case of  $u \neq 0, v = 0$  and  $u = v \neq 0$  can be proved similarly.

Next, we suppose that  $u \neq 0, v \neq 0$ , and  $u \neq v$ . To eliminate  $y^{2^i}$ , we add equation (29) multiplied by  $u$  to equation (30) multiplied by  $v$  and get

$$y = \frac{1}{C_2}(C_1x^{2^i} + C_3x + C_1u^{2^i}),$$

where

$$\begin{aligned} C_1 &= u^2 + uv + v^2, \\ C_2 &= u^{2^i}v + uv^{2^i}, \\ C_3 &= u^{2^i+1} + u^{2^i}v + v^{2^i+1}. \end{aligned}$$

It is easy to see that  $C_1 \neq 0$  and  $C_2 \neq 0$  since that  $k$  is odd,  $\gcd(i, k) = 1, u \neq 0, v \neq 0$ , and  $u \neq v$ . Substitute the above equation to equation Eq.(29) and multiply both sides by  $(C_2)^{2^i+1}$ , then we obtain

$$\begin{aligned} & vC_2C_1^{2^i}x^{2^{2i}} + \left(uC_2^{2^i+1} + (vC_2)^{2^i}C_1 + vC_2C_3^{2^i}\right)x^{2^i} + \left(u^{2^i}C_2^{2^i+1} + (vC_2)^{2^i}C_3\right)x \\ &= vC_2C_1^{2^i}u^{2^{2i}} + (vC_2)^{2^i}C_1u^{2^i} + C_2^{2^i+1}(u^{2^i+1} + v^{2^i+1}). \end{aligned}$$

We simplify respectively the coefficient of each term of the above equation, and finally get

$$C_2 x^{2^{2i}} + (u^{2^{2i}} v + uv^{2^{2i}}) x^{2^i} + C_2^{2^i} x = u^{2^{2i}} C_2 + u C_2^{2^i}.$$

Divide both sides by  $u^{2^{2i}+2^i+1}$ , then we have

$$\begin{aligned} & \left( \frac{v}{u} + \left( \frac{v}{u} \right)^{2^i} \right) \left( \frac{x}{u} \right)^{2^{2i}} + \left( \frac{v}{u} + \left( \frac{v}{u} \right)^{2^{2i}} \right) \left( \frac{x}{u} \right)^{2^i} + \left( \left( \frac{v}{u} \right)^{2^i} + \left( \frac{v}{u} \right)^{2^{2i}} \right) \frac{x}{u} \\ &= \frac{v}{u} + \left( \frac{v}{u} \right)^{2^i} + \left( \frac{v}{u} \right)^{2^i} + \left( \frac{v}{u} \right)^{2^{2i}}. \end{aligned} \quad (31)$$

Denote  $w = \frac{v}{u} + \left( \frac{v}{u} \right)^{2^i}$ ,  $z = \frac{x}{u} + \left( \frac{x}{u} \right)^{2^i}$ . Then  $w \neq 0$  since  $u \neq 0, v \neq 0$  and  $u \neq v$ . The above equation is equivalent to

$$c(z+1)^{2^i} + c^{2^i}(z+1) = 0. \quad (32)$$

The solution of Eq.(32) is  $z = 1$  or  $z = w + 1$  because  $\gcd(i, k) = 1$ .

If  $z = 1$ , i.e.,  $\frac{x}{u} + \left( \frac{x}{u} \right)^{2^i} = 1$ . In this case, Eq.(31) has no solution over  $\mathbb{F}_{2^k}$ . Otherwise, we have  $\text{Tr}\left(\frac{x}{u} + \left( \frac{x}{u} \right)^{2^i}\right) = 0 \neq \text{Tr}(1)$  since  $k$  is odd.

If  $z = w + 1$ , i.e.,  $\frac{x}{u} + \left( \frac{x}{u} \right)^{2^i} = \frac{v}{u} + \left( \frac{v}{u} \right)^{2^i} + 1$ . In this case, Eq.(31) has no solution over  $\mathbb{F}_{2^k}$  as well.

This completes the proof.  $\square$

**Remark 3.** We have also studied experimentally the bijective property of the closed butterfly structure with other  $\alpha$ . However, we could not find an  $\alpha \neq 0, 1$  such that  $V_e^\alpha$  is a permutation over  $\mathbb{F}_{2^k}$ . We conjecture that  $V_e^\alpha$  is a permutation over  $\mathbb{F}_{2^k}$  if and only if  $\alpha = 1$ .

#### 4.2. Other Cryptographic Properties

When  $\alpha = 1$  in the butterfly structure, the  $H_e^1$  is functionally equivalent to the 3-round Feistel structure constructed by Li and Wang [13]. They proved the differential spectrum is  $\{0, 4\}$  and the algebraic degree is  $k$ . Next we consider the nonlinearity. Firstly, we need the following results.

**Lemma 13 ([13]).** Suppose  $k$  is an odd integer and  $\gcd(i, k) = 1$ . Then for any  $(c, d) \in \mathbb{F}_{2^k}^2$  with  $(c, d) \neq (0, 0)$ , the following system of equations in  $x, y$

$$\begin{cases} d^{2^i} x + (dx)^{2^{k-i}} + (c+d)^{2^i} y + ((c+d)y)^{2^{k-i}} = 0, \\ c^{2^i} x + (cx)^{2^{k-i}} + d^{2^i} y + (dy)^{2^{k-i}} = 0, \end{cases} \quad (33)$$

has exactly 4 solutions over  $\mathbb{F}_{2^k}$ .

We call  $(x, y)$  nonzero if  $(x, y) \neq (0, 0)$ . Note that  $(0, 0)$  is always a solution of Eq.(33). So for any  $(c, d) \in \mathbb{F}_{2^k}^2$  with  $(c, d) \neq (0, 0)$ , Eq.(33) has exactly 3 nonzero solutions over  $\mathbb{F}_{2^k}$ . One can easily verify that the three nonzero solutions are  $(x, y)$ ,  $(y, x + y)$  and  $(x + y, x)$  if  $(x, y)$  is one of nonzero solutions of Eq.(33).



Denote  $S_{(a,b)} = \{(a,b), (b,a+b), (a+b,a)\}$ . Obviously, any one element in  $S_{(a,b)}$  determines completely the set, i.e.,  $S_{(a,b)} = S_{(b,a+b)} = S_{(a+b,a)}$ . Furthermore for any  $(a,b) \neq (0,0), (c,d) \neq (0,0)$ , either we have  $S_{(a,b)} = S_{(c,d)}$ , or we have  $S_{(a,b)} \cap S_{(c,d)} = \emptyset$ . Put

$$\mathcal{S} = \{S_{(a,b)} : (a,b) \in \mathbb{F}_{2^k}^2, (a,b) \neq (0,0)\},$$

then obviously  $\mathcal{S}$  is finite.

**Lemma 14.** *Suppose  $k$  is an odd integer and  $\gcd(i,k) = 1$ . Then for any  $(c,d) \in \mathbb{F}_{2^k}^2$  with  $(c,d) \neq (0,0)$ , the following system of equations in variables  $u$  and  $v$*

$$\begin{cases} du^{2^i} + (du)^{2^{k-i}} + (c+d)v^{2^i} + ((c+d)v)^{2^{k-i}} = 0, \\ (c+d)u^{2^i} + ((c+d)u)^{2^{k-i}} + cv^{2^i} + (cv)^{2^{k-i}} = 0 \end{cases} \quad (34)$$

has exactly 4 solutions over  $\mathbb{F}_{2^k}$ .

PROOF. Firstly, we show that Eq.(34) has at most 4 solutions. We add the first equation to the second equation and obtain

$$\begin{cases} du^{2^i} + (du)^{2^{k-i}} + (c+d)v^{2^i} + ((c+d)v)^{2^{k-i}} = 0, \\ cu^{2^i} + (cu)^{2^{k-i}} + dv^{2^i} + (dv)^{2^{k-i}} = 0, \end{cases} \quad (35)$$

then raise both equations to the  $2^i$ th power, we have

$$\begin{cases} d^{2^i}u^{2^{2i}} + du + (c+d)^{2^i}v^{2^{2i}} + (c+d)v = 0, \\ c^{2^i}u^{2^{2i}} + cu + d^{2^i}v^{2^{2i}} + dv = 0. \end{cases} \quad (36)$$

If  $c = 0$ , then  $d \neq 0$ , Eq.(37) in  $v$  has 2 solutions. For each solution  $y$ , Eq.(36) in  $u$  has at most 2 solutions. Hence, Eq.(34) has at most 4 solutions. The cases  $d = 0, c \neq 0$  and  $c = d \neq 0$  is identical.

Next, we suppose that  $u \neq 0, v \neq 0$ , and  $u \neq v$ . We add Eq.(36) multiplied by  $c^{2^i}$  to Eq.(37) multiplied by  $d^{2^i}$  to eliminate  $u^{2^{2i}}$ , then replace the  $u$  into Eq.(36) and get  $D_1v^{2^{4i}} + D_2v^{2^{2i}} + D_3v = 0$ , where  $D_1 = d^{2^i} \frac{(c^2+cd+d^2)^{2^{3i}}}{(c^{2^i}d+cd^{2^i})^{2^i}} \neq 0$ . According to Corollary 4, this equation in  $v$  has at most 4 solutions. Since the solution  $u$  is uniquely determined by  $v$ , Eq.(34) has at most 4 solutions.

Considering the following system of equations

$$\begin{cases} U^{2^i}X + (UX)^{2^{k-i}} + (U+V)^{2^i}Y + ((U+V)Y)^{2^{k-i}} = 0, \\ V^{2^i}X + (VX)^{2^{k-i}} + U^{2^i}Y + (UY)^{2^{k-i}} = 0, \end{cases} \quad (38)$$

If we fix  $(U,V) = (d,c) \neq (0,0)$ , then Eq.(38) with variables  $X$  and  $Y$  has exactly 3 nonzero solutions over  $\mathbb{F}_{2^k}^2$  from Lemma 13. W.l.o.g., suppose  $(x,y), (y,x+y)$  and  $(x+y,x)$  are the three nonzero solutions. If we fix  $(U,V) = (c,c+d)$  or  $(U,V) = (c+d,d)$ , it is easy to verify that the three nonzero solutions of Eq.(38) in  $X$  and  $Y$  are also  $(x,y), (y,x+y)$  and  $(x+y,x)$ .

Define a map

$$\begin{aligned}\phi: \quad \mathcal{S} &\longrightarrow \mathcal{S} \\ S_{(a,d)} &\longmapsto S_{(x,y)},\end{aligned}$$

where  $(x, y)$  is any nonzero solution of Eq.(38) in variables  $X$  and  $Y$  with respect to coefficients  $(U, V) = (a, b)$ . This map is well-defined from above illustration. Then according to what we have showed, Eq.(35) has at most 4 solutions, which means  $\phi$  is injective. But since  $\mathcal{S}$  is finite, therefore  $\phi$  is bijective. So for any  $(d, c) \in \mathbb{F}_{2^k}^2, (d, c) \neq (0, 0)$ , there exists  $(u, v) \neq (0, 0)$  such that  $\phi(S_{(u,v)}) = S_{(d,c)}$ , which is mean that if  $(X, Y) = (d, c)$ , then Eq.(38) in variable  $U$  and  $V$  has exactly three nonzero solutions.

We complete the proof.  $\square$

The proof of nonlinearity is completely identical to the proof in Theorem 11. From Lemma 14, we have  $m = 2$ , the  $\mathcal{W}_F^2((a, b), (c, d)) = 0$ , or  $2^{2k+2}$ . Therefore,  $\Lambda_F = \{0, \pm 2^{k+1}\}$ , and the nonlinearity  $\mathcal{NL}(F) = 2^{2k-1} - 2^k$ .

At the end of this section, we summarize the main results as follows.

**Theorem 15.** *Suppose  $k$  is an odd integer and  $\gcd(i, k) = 1$ . For any  $0 \leq t \leq k - 1$ , let  $H_e^1$  and  $V_e^1$  be the open and closed  $2k$ -bit butterflies structure with exponent  $e = (2^i + 1) \times 2^t$ . then*

- (1) *Both of  $H_e^1$  and  $V_e^1$  are permutation over  $\mathbb{F}_{2^k}^2$ .*
- (2) *The algebraic degree of  $H_e^1$  and  $V_e^1$  are equal to, respectively,  $k$  and  $2$ .*
- (3) *The differential uniformity of both  $H_e^1$  and  $V_e^1$  are equal to  $4$  and the differential spectrum are  $\{0, 4\}$ .*
- (4) *The nonlinearity of both  $H_e^1$  and  $V_e^1$  are equal to  $2^{2k-1} - 2^k$ , namely, optimal, and their Walsh spectrum are  $\{0, \pm 2^{k+1}\}$ .*

## 5. Conclusion

In the present paper, we further study the butterfly structure and show that these structure always have very good cryptographic properties. Moreover, we prove the nonlinearity is optimal in the general case. The research of finding more classes of differentially 4-uniform permutations with highly nonlinearity and algebraic degree from other functions over subfields is very interesting and is worthy of a further investigation. The following questions is still open.

**Open Problems:** Is there a tuple  $k, e, \alpha$  where  $k > 3$  and  $e$  are integers, and  $\alpha$  is a finite field element such that  $H_e^\alpha$  operating on  $\mathbb{F}_{2^k}^2$  is APN?

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## Appendix

### The proof of $B_7 \neq 0$ :

Otherwise, we suppose that  $B_7 = 0$ . For simplify expressions, denote  $\beta = \alpha^{2^i}$ , we have

$$(\beta c + \alpha d)B_6^{2^i} = (\alpha\beta d + c + d)B_4^{2^i}.$$

Replace the above equation by the expressions  $B_4$  and  $B_6$ , we have the following expression

$$\begin{aligned} & (\beta c + \alpha d)^{2^{2i}} \left[ (\beta c + \alpha d)(\beta c + \alpha d)^{2^i} + (\alpha\beta d + c + d)(\alpha\beta c + c + d)^{2^i} \right] \\ &= (\alpha\beta c + c + d)^{2^{2i}} \left[ (\beta c + \alpha d)(\alpha\beta d + c + d)^{2^i} + (\alpha\beta d + c + d)(\alpha c + \beta d)^{2^i} \right]. \end{aligned}$$

Note that  $\alpha\beta c + c + d \neq 0$  and  $\beta c + \alpha d \neq 0$ , we have

$$(\beta c + \alpha d)(\alpha\beta d + c + d)^{2^i} + (\alpha\beta d + c + d)(\alpha c + \beta d)^{2^i} = \frac{(\beta c + \alpha d)^{2^{2i}}}{(\alpha\beta c + c + d)^{2^{2i}}} B_6. \quad (39)$$

We also have  $\alpha\beta d + c + d \neq 0$ , otherwise, we must have  $(\beta c + \alpha d)B_6^{2^i} = 0$ , which is impossible. From  $B_5 = 0$ , we get

$$\begin{aligned} & (\alpha\beta d + c + d)(\beta c + \alpha d)(\alpha c + \beta d)^{2^i}(\beta c + \alpha d)^{2^i} \\ &= (\alpha\beta d + c + d)(\beta c + \alpha d)(\alpha\beta c + c + d)^{2^i}(\alpha\beta d + c + d)^{2^i}. \end{aligned}$$

Replace  $(\beta c + \alpha d)(\beta c + \alpha d)^{2^i} = B_6 + (\alpha\beta c + c + d)^{2^i}(\alpha\beta d + c + d)$  into the above equation and obtain

$$\begin{aligned} & B_6(\alpha\beta d + c + d)(\alpha c + \beta d)^{2^i} \\ &= (\alpha\beta c + c + d)^{2^i}(\alpha\beta d + c + d) \left[ (\beta c + \alpha d)(\alpha\beta d + c + d)^{2^i} + (\alpha\beta d + c + d)(\alpha c + \beta d)^{2^i} \right]. \end{aligned}$$

According to Eq.(39) and  $B_6 \neq 0$ , we deduce that

$$(\alpha\beta d + c + d)(\alpha c + \beta d)^{2^i} = (\alpha\beta c + c + d)^{2^i}(\alpha\beta d + c + d) \frac{(\beta c + \alpha d)^{2^{2i}}}{(\alpha\beta c + c + d)^{2^{2i}}},$$

which is equivalent to

$$(\alpha c + \beta d)(\alpha\beta c + c + d)^{2^i} = (\alpha\beta c + c + d)(\beta c + \alpha d)^{2^i},$$

meaning  $B_4 = 0$ , a contradiction. Hence, we complete the proof.  $\square$